# Introduction to Mathematical Quantum Theory

**Solution to the Exercises**

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**Exercise 1**

Let $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Recall that in class we proved

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}}\,\widehat{f}\widehat{g}. \tag{1}$$

Prove that

$$\widehat{f} * \widehat{g} = (2\pi)^{\frac{d}{2}}\,\widehat{fg}. \tag{2}$$

*Hint: Consider the equivalent statement of* (1) *for the inverse of the Fourier transform and apply it to* $\widehat{fg}$.

*Proof.* Recall that the inverse Fourier transform is such that $\check{f}\left(\mathbf{x}\right) = \widehat{f}\left(-\mathbf{x}\right)$. We then use (1) to get

$$\widecheck{f * g}\left(\mathbf{x}\right) = \widehat{f * g}\left(-\mathbf{x}\right) = (2\pi)^{\frac{d}{2}}\,\widehat{f}\left(-\mathbf{x}\right)\widehat{g}\left(-\mathbf{x}\right) = (2\pi)^{\frac{d}{2}}\,\check{f}\left(\mathbf{x}\right)\check{g}\left(\mathbf{x}\right).$$

To prove (2), consider $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Then we know that

$$\check{\widehat{f}} = f, \qquad \check{\widehat{g}} = g.$$

Given that $fg \in \mathscr{S}\left(\mathbb{R}^d\right)$, using the formula we deduced for the inverse Fourier transform, we can get

$$\widehat{fg} = \widehat{\check{\widehat{f}}\check{\widehat{g}}} = (2\pi)^{-\frac{d}{2}}\,\widehat{\widecheck{\widehat{f} * \widehat{g}}} = (2\pi)^{-\frac{d}{2}}\,\widehat{f} * \widehat{g}.$$

□

**Exercise 2**

Let $\mathcal{H}$ be an Hilbert space and $V$ a closed linear subspace of $\mathcal{H}$.

**a** In class we proved that for any $f \in \mathcal{H}$ there exists an element $g_f \in V$ such that

$$\left\|f - g_f\right\| = \min_{h\in V}\left\|f - h\right\|. \tag{3}$$

Prove that $g_f$ is the unique element of $V$ that satisfies the minimum.

**b** In class we proved that $g_f$ is such that $f - g_f \in V^\perp$. Prove that there is no other element $h \in V$ such that $f - h \in V^\perp$.

*Proof.* Recall the parallelogram law; for any $f,\ g \in \mathcal{H}$ we have

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2 . \tag{4}$$

To prove **a**, consider $g' \in$ such that

$$\|f - g'\| = \min_{h \in V} \|f-h\| .$$

This implies in particular that $\|f-g'\| = \|f-g_f\|$. By parallelogram law we deduce that

$$\begin{aligned}\|g_f - g'\|^2 &= \|(f-g') - (f-g_f)\|^2 \\ &= 2\|f-g'\|^2 + 2\|f-g_f\|^2 - \|(f-g') + (f-g_f)\|^2 \\ &= 4\|f-g_f\|^2 - \|2f - (g'+g_f)\|^2 = 4\|f-g_f\|^2 - 4\left\|f - \frac{1}{2}(g'+g_f)\right\|^2 .\end{aligned}$$

Given that $V$ is a vector space, we get that $\frac{1}{2}(g'+g_f) \in V$, and therefore

$$\|g_f - g'\|^2 \leqslant 4\|f-g_f\|^2 - 4\inf_{h \in V}\|f-h\|^2 = 0$$

and therefore $g' = g_f$.

To prove **b** suppose that $g' \in V$ is such that $f - g' \in V^\perp$. Then by definition of $V^\perp$, for any $h \in V$ we get

$$\langle h, g_f - g' \rangle = \langle h, f-g' \rangle - \langle h, f - g_f \rangle = 0,$$

where the last equality comes from the fact that both $f - g_f$ and $f - g'$ are in $V^\perp$. We therefore have that $g_f - g' \in V^\perp$. At the same time, $g_f - g' \in V$, and this implies $g' = g_f$.

□

### Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Prove that there exists a basis for $\mathcal{H}$. Prove moreover that $\mathcal{H}$ is separable if and only if there exists a countable base for it.

*Hint: For the first part apply Zorn's Lemma to the set of (also infinite) orthonormal systems ordered by inclusion. Prove that any maximal orthonormal system is a base, i.e. is dense.*

*For the second part prove and use the following fact: if $f$ is an element of $\mathcal{H}$ and $S$ is a basis for $\mathcal{H}$, there exists a sequence of elements $\{e_n\}_{n\in\mathbb{N}} \subseteq S$ such that $f \in \overline{\operatorname{span}_{\mathbb{K}}\{e_n\}_{n\in\mathbb{N}}}$.*

*Proof.* To prove the first part, call $\mathcal{A}$ the set of all orthonormal systems in $\mathcal{H}$, i.e.

$$\mathcal{A} := \left\{S \subseteq \mathcal{H} : \ \langle \psi, \psi' \rangle = \delta_{\psi,\psi'} \ \forall \psi, \psi' \in S\right\},$$

where $\delta_{\psi,\psi'}$ is 1 if $\psi = \psi'$ and 0 otherwise.

Consider then the set $\mathcal{A}$ with the partial order given by the inclusion. To apply Zorn's Lemma consider $\mathcal{B}$ an inductive ordered subset of $\mathcal{A}$. Consider moreover

$$S_{\mathcal{B}} := \bigcup_{S \in \mathcal{B}} S.$$

We want to prove that this is an upper bound for $\mathcal{B}$.

First we prove that $S_{\mathcal{B}} \in \mathcal{A}$. Given that $\mathcal{H}$ is closed with respect to unions, $S_{\mathcal{B}} \subseteq \mathcal{H}$. Let now $\psi,\ \psi' \in S_{\mathcal{B}}$; then there exist two orthonormal systems $S,\ S'$ such that $\psi \in S \in \mathcal{B}$ and $\psi' \in S' \in \mathcal{B}$; given that $\mathcal{B}$ is ordered, either $S \subseteq S'$ or $S' \subseteq S$. Suppose $S \subseteq S'$; then $\psi, \psi' \in S'$ and we get that $\langle \psi, \psi' \rangle = \delta_{\psi,\psi'}$, and therefore $S_{\mathcal{B}} \in \mathcal{A}$. Given that for any $S \in \mathcal{B}$ we have $S \subseteq S_{\mathcal{B}}$, this is clearly an upper bound for $\mathcal{B}$.

We can now apply Zorn's Lemma to deduce the existence of a maximal element of $\mathcal{A}$. What is left to prove is that this maximal element is a basis. Call $S$ the maximal element; by definition this is an orthonormal system. We prove that it is dense. Suppose it is not; then[1] $V := \overline{\mathrm{span}_{\mathbb{K}}\{S\}}$ is a well defined closed vector space such that $\mathcal{H} \backslash V \neq \varnothing$ and $V^{\perp}$ is nonempty. Let now $\phi \in V^{\perp}$ such that $\|\phi\| = 1$ and let $S_{\phi} := \{\phi\} \cup S$. Clearly $S_{\phi} \subseteq \mathcal{H}$; consider $\psi,\ \psi' \in S_{\phi}$; if $\psi \neq \phi \neq \psi'$ from the fact that $S$ is an orthonormal system we already know that $\langle \psi, \psi' \rangle = \delta_{\psi,\psi'}$. Suppose now $\psi \in S$; given that $\phi \in V^{\perp}$ we get $\langle \psi, \phi \rangle = 0$. Given that $\langle \phi, \phi \rangle = \|\phi\|^2 = 1$ we deduce that $S_{\psi}$ is an orthonormal system. But now $S_{\psi} \supseteq S$ and $S_{\psi} \neq S$, which contradicts the maximality of $S$. Therefore $S$ is a basis for $\mathcal{H}$.

To prove the second part, we first prove the fact in the hint. Indeed, $f \in \mathcal{H}$ implies that there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \to f$ and

$$f_n = \sum_{j=1}^{N(n)} a_{j,n} e_{j,n}$$

for some $N(n) \in \mathbb{N}$, $\{a_{j,n}\}_{j,n \in \mathbb{N}} \subseteq \mathbb{K}$ and $\{e_{j,n}\}_{j,n \in \mathbb{N}} \subseteq S$. Given that the latter is a countable sequence in $S$ this proves the fact.

Now, we use this fact to prove our Exercise; suppose that $\mathcal{H}$ is separable; therefore, there exists $D$ a dense subset of $\mathcal{H}$ which is countable, i.e., $D = \{d_n\}_{n \in \mathbb{N}}$. But for every $n \in \mathbb{N}$, $d_n$ is in the span of $S_n$ a countable subset of $S$; we then get the following chain of inequalities:

$$\mathcal{H} = \overline{D} = \overline{\{d_n\}_{n \in \mathbb{N}}} \subseteq \overline{\bigcup_{n \in \mathbb{N}} S_n} \subseteq \overline{S} = \mathcal{H},$$

and the inequalities are in fact equalities.

From this we get that $\bigcup_{n \in \mathbb{N}} S_n$ is dense in $\mathcal{H}$ and given that $\bigcup_{n \in \mathbb{N}} S_n \subset S$, this is also an orthonormal system, therefore is a basis. Moreover, it is union of countable sets, so it is also countable, and this proves the first implication.

Suppose now that $S$ is a countable basis for $\mathcal{H}$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$. Call then $\mathbb{F}$ a countable dense subset of $\mathbb{K}$ We have that $D := \mathrm{span}_{\mathbb{F}}\{S\}$ is countable and dense in $\mathcal{H}$.

[1] We indicate with $\mathrm{span}_{\mathbb{K}}\{A\}$ the set of finite linear combinations of elements in $A$ with coefficients in $\mathbb{K}$.

□

**Exercise 4**

Let $A$, $B$ bounded operators on an Hilbert space $\mathcal{H}$ and $\alpha$, $\beta \in \mathbb{C}$. Prove the following equalities:

$$\mathrm{id}^* = \mathrm{id} \tag{5}$$
$$(A^*)^* = A \tag{6}$$
$$(AB)^* = B^* A^* \tag{7}$$
$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*. \tag{8}$$

Moreover, prove that $A^*$ is bounded and that $\|A^*\| = \|A\|$.

*Proof.* For the proof of (5) consider $\psi \in \mathcal{H}$. For any $\phi \in \mathcal{H}$ from the definition of the adjoint we get

$$\langle \phi, \mathrm{id}^* \psi \rangle = \langle \mathrm{id}\, \phi, \psi \rangle = \langle \phi, \psi \rangle \Rightarrow \langle \phi, \mathrm{id}^* \psi - \psi \rangle = 0,$$

and by density we can imply that $\mathrm{id}^* \psi = \psi$.

For the proof of (6) we get that for any $\phi$, $\psi \in \mathcal{H}$

$$\langle \phi, (A^*)^* \psi \rangle = \langle A^* \phi, \psi \rangle = \overline{\langle \psi, A^* \phi \rangle} = \overline{\langle A\psi, \phi \rangle} = \langle \phi, A\psi \rangle.$$

Analogously as before we conclude by density that $(A^*)^* = A$.

For the proof of (7) we get that for any $\phi$, $\psi \in \mathcal{H}$

$$\langle \phi, (AB)^* \psi \rangle = \langle AB\phi, \psi \rangle = \langle B\phi, A^*\psi \rangle = \langle \phi, B^* A^* \psi \rangle.$$

For the proof of (8) we get that for any $\phi$, $\psi \in \mathcal{H}$

$$\begin{aligned} \langle \phi, (\alpha A + \beta B)^* \psi \rangle &= \langle (\alpha A + \beta B)\, \phi, \psi \rangle = \overline{\alpha} \langle A\phi, \psi \rangle + \overline{\beta} \langle B\phi, \psi \rangle \\ &= \overline{\alpha} \langle \phi, A^*\psi \rangle + \overline{\beta} \langle \phi, B^*\psi \rangle = \langle \phi, \left( \overline{\alpha} A^* + \overline{\beta} B^* \right) \psi \rangle, \end{aligned}$$

and we can conclude again by density.

To see that $A^*$ is bounded consider $\psi \in \mathcal{H}$; then we have

$$\|A^*\psi\|^2 = \langle A^*\psi, A^*\psi \rangle = \langle \psi, AA^*\psi \rangle \leqslant \|\psi\| \, \|AA^*\psi\| \leqslant \|\psi\| \, \|A\| \, \|A^*\psi\| \,,$$

and therefore we get $\|A^*\psi\| \leqslant \|A\| \, \|\psi\|$; as a consequence we get $\|A^*\| \leqslant \|A\|$, and therefore

$$\|A^*\| \leqslant \|A\| = \left\|(A^*)^*\right\| \leqslant \|A^*\| \,,$$

and hence $\|A^*\| = \|A\|$.

□